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FORMULATION OF SOLUTIONS OFSTANDARD BI-QUADRATIC CONGRUENCE OF EVEN COMPOSITE MODULUS- AN EIGHTH MULTIPLE OF A POWERED ODD PRIME IN SOME SPECIAL CASES.

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ABSTRACT

In this paper, the author has established a formulation is to find the solutions of standard bi-quadratic congruence of even composite modulus-an eighth multiple of a powered odd prime in some special cases. The formula discovered is verified true by solving some examples. The formula works wonderfully.Formulation of solutions is the merit of the paper.

KEY-WORDS

Bi-quadratic congruence, Binomial expansion formula, Chinese Remainder Theorem.

INTRODUCTION

Astandard bi-quadratic congruence is a congruence of fourth degree of the type:

$\boldsymbol{x^4 \equiv a^4 (mod\ p); p\ an\ odd\ positive\ prime\ integer}$,**is called a standard bi-quadratic congruence of prime modulus. Such type of congruence are always solvable. A little material about the solving of bi-quadratic congruence is available. The author already has formulated some classes of standard bi-quadratic congruence of composite modulus.**

PROBLEM-STATEMENT

The problem of study is-

"To establish a formula for the solutions of the standard bi-quadratic congruence:

$$(1) x^4 \equiv a^4 (mod\ 8.p^n);$$

$(2) x^4 \equiv p^4 (mod\ 8.p^n);$ p being a positive prime integer; n any positive integer.

LITERATURE REVIEW

The author referred many books of Number theory [1], [2], [3] andfound aninsufficient discussion on standard bi-quadratic congruence with no formulation. Only Zukerman and

Koshy had discussed a bit in their books of Number Theory. Only the author's formulations are found [4], [5], [6], [7].

NEED OF RESEARCH

The literature of mathematics says approximately nothing about the said standard bi-quadratic congruence. Some discussion on general bi-quadratic congruence is found. The bi-quadratic congruence under consideration can be solved by a time-consuming and complicated method, known as Chinese Remainder Theorem (CRT) [1]. Readers do not want to use the CRT for solutions. The author tried his best with sincere effort to formulate some more congruence and presented the result in this paper. This is the need of the research.

ANALYSIS & RESULTS

Case-I: When $\boldsymbol{a} \neq \boldsymbol{p}$.

Consider the congruence: $x^4 \equiv a^4 (mod\ 8p^n); p\ being\ a\ positive\ prime\ integer$.

If $x \equiv 2p^n k \pm a\ (mod\ 8p^n)$, then by binomial expansion formula

$$x^4 \equiv (2p^n k \pm a)^4 (mod\ 8p^n)$$

$$\equiv (2p^n k)^4 + 4.(2p^n k)^3.a + \frac{4.3}{1.2}(2p^n k)^2.a^2 + \frac{4.3.2}{1.2.3}(2p^n k)^1.a^3 + a^4 (mod\ 8p^n),$$

$$\equiv 8p^n(\dots\dots) + a^4\ (mod\ 8p^n)$$

$$\equiv a^4\ (mod\ 8p^n).$$

Therefore, $x \equiv 2p^n k \pm a\ (mod\ 8p^n)$ satisfies the congruence $x^4 \equiv a^4 (mod\ 8p^n)$ and hence can be considered as solutions formula of the said congruence.

But for $k = 4$, the formula reduces to $x \equiv 2p^n.4 \pm a \equiv 8p^n \pm a \equiv 0 \pm a\ (mod\ 8p^n)$.

This is the same solutions as for $k = 0$.

Also, for $k = 5 = 4 + 1$, it is easily seen that the solutions are the same as for k=1.

Hence it can be concluded that the congruence has exactly eight incongruent solutions

$$x \equiv 2p^n k \pm a\ (mod\ 8p^n) with\ k = 0, 1, 2, 3.$$

Case-II: When $\boldsymbol{a} = \boldsymbol{p}$ & $n = 2$.

Then the congruence reduces to: $x^4 \equiv p^4 (mod\ 8p^2); p\ being\ a\ positive\ prime\ integer$.

If $x \equiv 2pk + p\ (mod\ 8p^2)$, then by binomial expansion formula

$$x^4 \equiv (2pk+p)^4 (mod\ 8p^2)$$

$$\equiv (2pk)^4 + 4.(2pk)^3.p + \frac{4.3}{1.2}(2pk)^2.p^2 + \frac{4.3.2}{1.2.3}(2pk)^1.p^3 + p^4 (mod\ 8p^2)$$

$$\equiv 8p^2(\dots\dots) + p^4 (mod\ 8p^2)$$

$$\equiv p^4\ (mod\ 8p^2).$$

Therefore, $x \equiv 2pk \pm p\ (mod\ 8p^2)$ satisfies the congruence $x^4 \equiv p^4(mod\ 8p^2)$ and hence it is a solution of the said congruence.

But for $k = 4p, x \equiv 2p.4p \pm p = 8p^2 \pm p \equiv 0 \pm p\ (mod\ 8p^2)$.

This is the same solutions as for $k = 0$.

Also, for $k = 4p + 1$, it is easily seen that the solutions are the same as for k=1.

Hence it can be concluded that the congruence has exactly 4p incongruent solutions

$$x \equiv 2pk \pm p\ (mod\ 8p^2) with\ k = 0, 1, 2, \dots\dots\dots, (4p-1).$$

Case-III: When $\boldsymbol{a} = \boldsymbol{p}$ & $n = 3$.

Then the congruence reduces to: $x^4 \equiv p^4(mod\ 8p^3); p\ being\ a\ positive\ prime\ integer.$

If $x \equiv 2pk + p\ (mod\ 8p^3)$, then by binomial expansion formula

$$x^4 \equiv (2pk+p)^4 (mod\ 8p^3)$$

$$\equiv (2pk)^4 + 4.(2pk)^3.p + \frac{4.3}{1.2}(2pk)^2.p^2 + \frac{4.3.2}{1.2.3}(2pk)^1.p^3 + p^4 (mod\ 8p^3)$$

$$\equiv 8p^3(\dots\dots) + p^4 (mod\ 8p^3)$$

$$\equiv p^4\ (mod\ 8p^3).$$

Therefore, $x \equiv 2pk \pm p\ (mod\ 8p^3)$ satisfies the congruence $x^4 \equiv p^4(mod\ 8p^3)$ and hence it is a solution of the said congruence.

But for $k = 4p^2, x \equiv 2p.4p^2 \pm p = 8p^3 \pm p \equiv 0 \pm p\ (mod\ 8p^3)$.

This is the same solutions as for $k = 0$.

Also, for $k = 4p^2 + 1$, it is easily seen that the solutions are the same as for k=1.

Hence it can be concluded that the congruence has exactly $4p^2$ incongruent solutions

$$x \equiv 2pk \pm p \ (mod\ 8p^3) with\ k = 0, 1, 2, \dots \dots \dots, (4p^2 - 1).$$

Case-IV: When $a = p$ & $n \geq 4$.

Then the congruence reduces to $x^4 \equiv p^4 (mod\ 8p^n)$.

As in above, it can be easily seen that for $x \equiv 2p^{n-3}k + p\ (mod\ 8p^n)$,

$$x^4 \equiv (2p^{n-3}k + p)^4$$

$$\equiv (2p^{n-3}k)^4 + 4.(2p^{n-3}k)^3.p + \frac{4.3}{1.2}(2p^{n-3}k)^2.p^2 + \frac{4.3.2}{1.2.3}(2p^{n-3}k)^1.p^3 + p^4$$

$$\equiv 8p^n(\dots\dots) + p^4$$

$$\equiv p^4\ (mod\ 8p^n).$$

Therefore, $x \equiv 2p^{n-3}k + p\ (mod\ 8p^n)$ are the solutions of the said congruence.

But for $k = 4p^3, x = 2p^{n-3}.4p^3 + p = 8p^n + p \equiv 0 + p\ (mod\ 8p^n)$.

This is the same solutions as for $k = 0$.

Also, for $k = 8p^3 + 1$, it is easily seen that the solutions are the same as for k=1.

Hence it can be concluded that the congruence has exactly $4p^3$ incongruent solutions

$$x \equiv 2p^{n-3}k + p\ (mod\ 8p^n) with\ k = 0, 1, 2, 3 \dots \dots \dots \dots, (4p^3 - 1).$$

Sometimes the congruence are given in the form:$x^4 \equiv b\ (mod\ 8p^n)$

In such cases, it can be written as:$x^4 \equiv b + k.8p^n = a^4 (mod\ 8p^n)$.

ILLUSTRATIONS

Example-1: Consider the congruence $x^4 \equiv 81 (mod\ 392)$.

It can be written as $x^4 \equiv 3^4 (mod\ 8.49)$ $i.e.$ $x^4 \equiv 3^4\ (mod\ 8.7^2)$

It is of the type $x^4 \equiv a^4 (mod\ 8.p^n)$ $with\ a = 3,\ p = 7,\ n = 2,\ a \neq p$.

It has exactly eight incongruent solutions given by

$$x \equiv 2p^n k \pm a\ \ (mod\ 8.p^n)\ with\ k = 0, 1, 2, 3.$$

$$\equiv 2.7^2 k \pm 3\ \ (mod\ 8.7^2)$$

$$\equiv 98k \pm 3\ \ (mod\ 392)$$

$$\equiv 0 \pm 3;\ 98 \pm 3; 196 \pm 3; 294 \pm 3\ \ (mod\ 392)$$

$$\equiv 3{,}389;\ 95{,}101; 193, 199; 291, 297\ (mod\ 392)$$

$$\equiv 3,95,101,193,199,291,297,389\ (mod\ 392).$$

These are the required eight solutions.

Example-2: Consider the congruence $x^4 \equiv 256(mod\ 392)$.

It can be written as $x^4 \equiv 4^4(mod\ 8.49)\ \ i.e.\ \ x^4 \equiv 4^4\ (mod\ 8.7^2)$

It is of the type $x^4 \equiv a^4(mod\ 8.p^n)\ \ with\ a = 4,\ \ p = 7,\ \ n = 2.$

It has exactly eight incongruent solutions given by

$$x \equiv 2p^n k \pm a\ \ (mod\ 8.p^n)\ with\ k = 0, 1, 2, 3.$$

$$\equiv 2.7^2 k \pm 4\ \ (mod\ 8.7^2)$$

$$\equiv 98k \pm 4\ \ (mod\ 8.7^2)$$

$$\equiv 0 \pm 4;\ \ 98 \pm 4; 196 \pm 4;\ \ 294 \pm 4\ \ (mod\ 392)$$

$$\equiv 4, 388;\ \ 94, 102; 192, 200; 290, 298\ (mod\ 392)$$

$$\equiv 4, 94, 102,192, 200, 290, 29, 388\ \ (mod\ 392).$$

These are the required eight solutions.

Example-3: Consider the congruence $x^4 \equiv 49\ (mod\ 392)$.

It can be written as $x^4 \equiv 49 + 6.392 = 2401 = 7^4(mod\ 8.49)\ \ i.e.\ \ x^4 \equiv 7^4\ (mod\ 8.7^2)$

It is of the type $x^4 \equiv p^4(mod\ 8.p^2)\ \ with\ a = 7,\ \ p = 7,\ \ a = p.$

It has exactly twenty eight incongruent solutions given by

$$x \equiv 2pk + p\ \ (mod\ 8.p^n)\ with\ k = 0, 1, 2, 3, \dots \dots \dots, (4.7 - 1)$$

$$\equiv 2.7k + 7\ \ (mod\ 8.7^2) with\ k = 0, 1, 2, 3, \dots \dots \dots 27.$$

$$\equiv 14k + 7\ \ (mod\ 8.7^2)$$

$$\equiv 0 + 7;\ \ 14 + 7; 28 + 7;\ \ 42 + 7, \dots \dots \dots, 378 + 7\ \ (mod\ 392)$$

$$\equiv 7, 21, 35, 49, \dots \dots \dots \dots, 385\ (mod\ 392)$$

These are the required twenty eight solutions.

Example-4: Consider the congruence $x^4 \equiv 625\ (mod\ 1000)$

It can be written as$x^4 \equiv 5^4\ (mod\ 8.5^3)$

It is of the type $x^4 \equiv p^4(mod\ 8p^n)\ with\ \ a = 5,\ \ p = 5, n = 3, a = p.$

It has exactly $4p^2$ incongruent solutions given by

$$x \equiv 2p^{n-2}k + p \ (mod\ 8p^n); k = 0, 1 \dots \dots \dots, (4p^2 - 1)$$
$$\equiv 2.5k + 5\ (mod\ 8.5^3); k = 0, 1 \dots \dots \dots \dots \dots \dots \dots (4.5^2 - 1)$$
$$\equiv 10k + 5\ (mod\ 1000); k = 0, 1, 2, 3, 4 \dots \dots \dots \dots \dots \dots \dots .., 99.$$
$$\equiv 0 + 5; 10 + 5; 20 + 5; 30 + 5; 40 + 5; 50 + 5; \dots \dots \dots \dots \dots ..990 + 5\ (mod\ 1000)$$
$$\equiv 5,15,25,35, 45; \dots \dots \dots \dots \dots \dots \dots ,995\ (mod\ 1000).$$

These are the one hundred incongruent solutions of the congruence.

Example-5: Consider the congruence $x^4 \equiv 2401\ (mod\ 19208)$

It can be written as$x^4 \equiv 2401 = 7^4\ (mod\ 8.7^4)$

It is of the type $x^4 \equiv p^4 (mod\ 8p^n)\ with\ \ a = 7,\ \ p = 7, n = 4, a = p.$

It has exactly $4p^3$ incongruent solutions given by

$$x \equiv 2p^{n-3}k + p\ (mod\ 8p^n); k = 0, 1 \dots \dots \dots, (4p^3 - 1)$$
$$\equiv 2.7k + 7\ (mod\ 8.7^4); k = 0, 1 \dots \dots \dots \dots \dots \dots \dots (4.7^3 - 1)$$
$$\equiv 14k + 7\ (mod\ 19208); k = 0, 1, 2, 3, 4 \dots \dots \dots \dots \dots \dots \dots .., (1372 - 1).$$
$$\equiv 0 + 7; 14 + 7; 28 + 7; 42 + 7; 56 + 7; 70 + 7; \dots \dots \dots ,19194 + 7\ (mod\ 19208)$$
$$\equiv 7,21,35, 49,63,77, \dots \dots \dots \dots \dots \dots \dots ,19201\ (mod\ 19208).$$

These are the nineteen thousand two hundred and eight solutions of the congruence.

Example-6: Consider the congruence $x^4 \equiv 625\ (mod\ 25000)$.

It can be written as $x^4 \equiv 5^4\ (mod\ 8.5^5)$

It is of the type $x^4 \equiv a^4 (mod\ 8.p^n)\ with\ \ a = 5, p = 5, n = 5,\ \ a = p.$

It has five hundred solutions given by

$x \equiv 2p^{n-3}k + a\ (mod\ 8.p^n)\ with\ k = 0, 1, 2, 3$ …………, $(4p^3 - 1)$

$$\equiv 2.5^{5-3}k + 5\ (mod\ 8.5^5); k = 0, 1, 2, 3, \dots \dots \dots \dots, (4.5^3 - 1)$$
$$\equiv 50k + 5\ (md\ 25000); k = 0, 1, 2, 3, \dots \dots \dots \dots \dots ..,499.$$
$$\equiv 0 + 5\ ; 50 + 5; 100 + 5; 150 + 5; 200 + 5; \dots \dots \dots \dots ..24950 + 5\ (mod\ 8.5^5)$$
$$\equiv 5, 55, 105, 155, 205, \dots \dots \dots \dots \dots ..,24955\ (mod 25000)$$

These are the required solutions.

CONCLUSION

Thus, it is concluded that the standard bi-quadratic congruence:

$x^4 \equiv a^4 \ (mod\ 8.p^n); a \neq p$ has exactly eight incongruent solutions given by:

$$x \equiv 2p^n k \pm a \ \ (mod\ 8p^n)\ with\ k = 0, 1, 2, 3\ and\ p\ an\ odd\ prime.$$

But for $n = 2, a = p$, the congruence reduces to $x^4 \equiv p^4\ (mod\ 8.p^2)$ and has exactly $4p$ incongruent solutions given by:

$x \equiv 2pk + p\ \ (mod\ 8.p^2)\ with\ k = 0, 1, 2, 3, \dots \dots \dots, (4.p - 1).$

But for $n = 3, a = p$, the congruence reduces to $x^4 \equiv p^4\ (mod\ 8.p^3)$ and has exactly $4p^2$ incongruent solutions given by: $x \equiv 2pk + p\ (mod\ 8p^3); k = 0, 1 \dots \dots \dots, (4p^2 - 1).$

But for $n \geq 4, a = p$, the congruence reduces to $x^4 \equiv p^4 (mod\ 8.p^n)$

and has exactly $4p^3$ incongruent solutions given by:

$$x \equiv 2p^{n-3}k + p\ (mod\ 8p^n); k = 0, 1 \dots \dots \dots, (4p^3 - 1).$$

The established formulae are tested by solving different examples.

MERIT OF THE PAPER

The author discovered a direct formulation of solutions of the standard bi-quadratic congruence. The congruence is not found formulated in the literature of mathematics. Formulation proved simple and time-saving. This is the merit of the paper.

REFERENCE

[1] Zuckerman H. S., Niven I., 2008, *An Introduction to the Theory of Numbers*, Wiley India, Fifth Indian edition, ISBN: 978-81-265-1811-1.

[2] David M Burton,2012,*Elementary Number Theory*, McGraw Hill education (Higher Education), Seventh Indian Edition, New Dehli, India, ISBN: 978-1-25-902576-1.

[3] Thomas Koshy, 2009, *Elementary Number Theory with Applications*, Academic Press, Second Edition, Indian print, New Dehli, India, ISBN: 978-81-312-1859-4.

[4] Roy B M, *Formulation of solutions of standard bi-quadratic congruence of composite modulus -an integer multiple of power of an odd prime*, International Journal of Engineering Science and Mathematics (IJESM)/(IJMRA), ISSN: 2320-0294, Vol-08, Issue-12, Dec-19.

[5]Roy B M,*Formulation of solutions of standard biquadratic congruence of even composite modulus*,(IJETRM), ISSN: 2456-9348, Vol-, Issue-, Nov-19.

[6] Roy B M, *Formulation of a Special Class of Solvable Standard Bi-quadratic Congruence of Composite Modulus- an Integer Multiple of Power of Prime*, (IJSDR), ISSN:2455-2631, Vol-04, Issue-3, March-19.

[7]Roy B M, *Formulation Solutions of a special standard bi-quadratic congruence- modulo a powered odd prime*, International Journal of Engineering Technology Research & Management (IJETRM), ISSN: 2456-9348, Vol-03, Issue-11, April-21.

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